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On the Distribution of Vertex Energy of Connected Non-Regular Non-Bipartite Integral Graphs with Maximum Vertex Degree Four

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Abstract

The energy of a vertex in a graph plays a very important role in terms of its contribution to the total energy of a graph, a significant graph invariant in the field of chemical graph theory. Particularly, the computation of vertex energies of various types of graphs in general and integral graphs has been carried out in recent studies. Working in this direction, we determine the vertex energy of all connected non-regular non-bipartite integral graphs with maximum vertex degree four in this article.

Keywords: Vertex energy, non-regular graph, non-bipartite graph, integral graph.

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1 Introduction

In mathematical chemistry, the notion of graph energy, first proposed by Gutman [6], has been a very significant graph invariant. A lot of research has been done on graph energy and its variations because of its strong correlation with the total π-electron energy of molecules, found using basic Hückel orbital calculations.

Let $G = (V, E)$ be a simple undirected graph of order n with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The adjacency matrix of G is defined as $A = A(G) = [a_{ij}]_{n\times n}$ such that a_{ij} is 1 if there is an edge between the corresponding vertices v_i and v_j and 0 otherwise. Further, if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, all real as A is real symmetric, then the energy $\mathcal{E}(G)$ is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The study of integral graphs was formally initiated by Harary et al. in [8]. As discussed in this article, a graph is said to be integral if its adjacency matrix has only integral eigenvalues. The number of integral graphs is infinite and can be found in various graphs of different classes and orders but identifying them is very difficult that makes the study of such graphs more interesting and challenging. Also, focusing on studying connected integral graphs is sufficient as the spectra of a disconnected graph is simply the union of that of its connected components. In this direction, different classes of integral graphs with reference to their intrinsic properties such as degree, regularity and diameter have been studied rigorously in literature. For related work on integral graphs and a detailed survey, we refer [3–5].

As discussed in [3], the spectrum of a non-regular integral graph with maximum degree four lies in the segment $[-3, 3]$. Further, as studied by Radosavljevic and Simic [9], there are just thirteen connected non-regular non-bipartite integral graphs having maximum vertex degree four, as shown in Figure 1.

The concept of the energy of a vertex of the graph G was introduced in the year 2018 by Arizmendi et al. [2]. It is defined as

$$\mathcal{E}_G(v_i) = |A|_{ii}, \quad i = 1, 2, \cdots, n,$$

where $|A| = (AA^*)^{\frac{1}{2}}$ and A^* is the conjugate transpose of A.

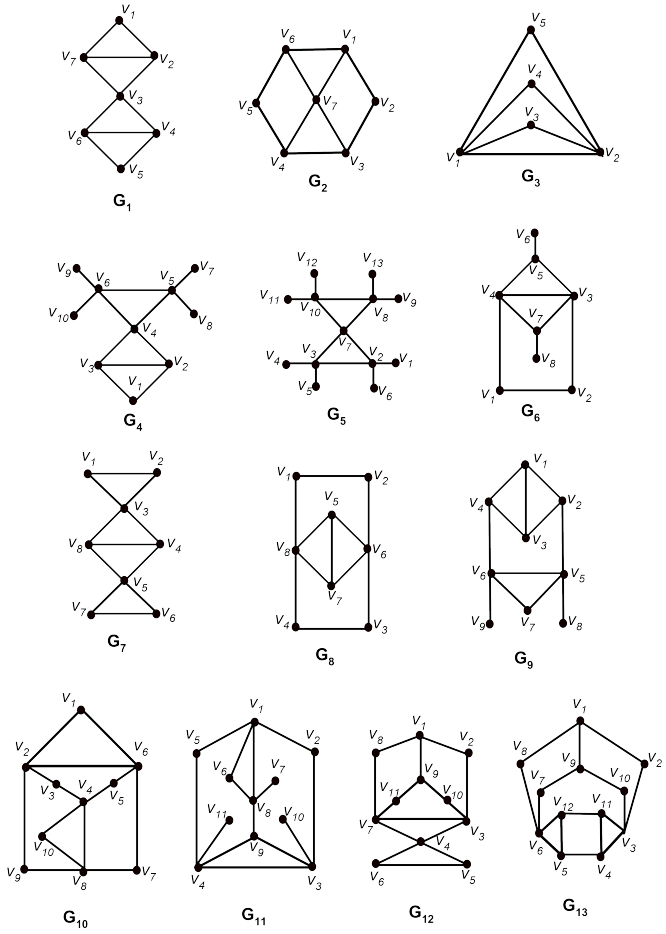

Figure 1: The thirteen connected non-regular non-bipartite integral graphs with maximum vertex degree four.

In [2], the authors showed that the energy of a graph is equal to the sum of the individual energies of all its vertices. Further, the authors in [1] presented an idea to compute the vertex energy in terms of the eigenvalues and eigenvectors and looking at the graph energy as the sum of the individual vertex energies in a graph. Additionally, they established a number of inequalities giving bounds for the vertex energy and discussed some examples and counterexamples of natural conjectures for this parameter in order to signify its use in future graph energy research. Later, Ramane et al. [10] computed the vertex energies of the subdivision of some of the

commonly studied graph structures, including complete graphs and complete bipartite graphs. Gutman and Furtula [7] introduced a method to calculate the energy of a vertex using the eigenvalue matrix and orthonormal matrix of the eigenvectors of the graph. In [11], Sharathkumar et al. determined the vertex energy of all non-trivial connected integral trees of order up to thirty. Similarly, Shrikanth et al. [12] worked on computing the vertex energies of all non-trivial connected integral graphs of order up to seven.

We now present some of the basic definitions and results, in literature, for further discussion.

Lemma 1. *[1] Let $G = (V, E)$ be a graph with $V(G) = \{v_1, \cdots, v_n\}$ and let $A = A(G) = [a_{ij}]$ be the adjacency matrix of G with $\lambda_1, \cdots, \lambda_n$ being the eigenvalues of A. Then, the energy of each vertex v_i is given by*

$$\mathcal{E}_G(v_i) = \sum_{j=1}^{n} p_{ij} \left|\lambda_j\right|, \quad i = 1, \cdots, n,$$

where λ_j denotes the j^{th} eigenvalue of A and the weights p_{ij}'s satisfy

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} p_{ij} = 1.$$

The values of p_{ij}'s can be obtained using the expression $p_{ij} = u_{ij}^2$ where $U = [u_{ij}]$ is the matrix representing the orthonormal eigenvectors corresponding to the eigenvalues of A.

Alternatively, as explained in [1], by considering the graph to be rooted at a given vertex, for each $k \in \mathbb{N}$, a measure $\phi_i(A^k)$ is associated to the graph by defining it as the k^{th} moment of A with respect to the linear functional ϕ_i. Then, we have the following lemma.

Lemma 2. *[1] Given a graph G of order n, let $\phi_i\left(A^k\right)$ be the k^{th} moment of A w. r. t. the linear functional ϕ_i, for $k \in \mathbb{N}$ and for each $i = 1, \cdots, n$. Then,*

$$\phi_i\left(A^k\right) = \sum_{j=1}^{n} p_{ij} \lambda_j^k$$

with $\phi_i\left(A^k\right)$ corresponding to the number of $v_i - v_i$ walks in G of length k.

Accordingly, for each i, by finding the number of $v_i - v_i$ walks in G of length k, we can construct a system of linear equations involving p_{ij}'s, that are obtained by solving the system.

Once p_{ij}'s are obtained using one of the above discussed methods, the energy of each vertex is obtained using their values in Lemma 1.

A graph G is said to be vertex-transitive, or node symmetric, if for any two vertices $u, v \in V(G)$, there exists an automorphism f such that $f(u) = v$. In other words, a vertex-transitive graph is the one in which every vertex is structurally identical to every other vertex so that no vertex is distinguishable from any other by means of the vertices and edges in its vicinity. Further, all vertex-transitive graphs are regular but not vice versa. Some common examples of vertex-transitive graphs are complete graphs, cycles, hypercubes and the famous Petersen graph.

Other than vertex-transitive graphs, there are graphs that have many vertex symmetries. For example, consider the graph G_1 in Fig. 1 with vertex set $V = \{v_1, \cdots, v_7\}$. It is easy to observe that it has three vertex symmetries in the form of $\{v_1, v_5\}, \{v_2, v_4, v_6, v_7\}$ and $\{v_3\}$. Similarly, in the graph G_4, there are five vertex symmetries, namely $\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5, v_6\}$ and $\{v_7, \cdots, v_{10}\}$.

During the study of vertex energy in a graph, instead of computing the energy of each vertex in it, one can compute the energy of one vertex in every symmetry set of its vertices, so that the energy of all the vertices in each such set remains the same. This method simplifies the whole process thereby reducing the complexity involved. Working in this direction, in the present study, we determine the vertex energy of all the thirteen connected non-regular non-bipartite integral graphs having maximum vertex degree four by observing the different vertex symmetries in them, and then using the method explained in [1, 11]. We use wxMaxima, a free, open-source, cross-platform GUI for the computer algebra system Maxima, to solve the system of linear equations involved in the study and to compute the vertex energies for each of these graphs.

2 Main Results

In this section, we determine the vertex energies of the 13 connected non-regular non-bipartite integral graphs having maximum vertex degree 4 given in Figure 1.

Theorem 3. *For the graph G_1, with $V(G_1) = \{v_1, \cdots, v_7\}$,*

$$\mathcal{E}_{G_1}(v) = \begin{cases} 1.0667 & \text{if } v = v_1, v_5, \\ 1.5667 & \text{if } v = v_2, v_4, v_6, v_7, \\ 1.6 & \text{otherwise.} \end{cases}$$

Proof. For the graph G_1, the characteristic polynomial is given by

$$P(G_1;\lambda) = -\lambda^7 + 10\lambda^5 + 8\lambda^4 - 21\lambda^3 - 32\lambda^2 - 12\lambda.$$

The eigenvalues of G_1 are $\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = -1$ and $\lambda_5 = 0$ with corresponding geometric multiplicities $[1, 1, 1, 3, 1]$. By observation, it is seen that the vertex set of G_1 can be partitioned into 3 sets, based on 3 different vertex symmetries, namely $V_1 = \{v_1, v_5\}$, $V_2 = \{v_2, v_4, v_6, v_7\}$ and $V_3 = \{v_3\}$. Accordingly, all the vertices belonging to a symmetry set have the same energy. Further, since there are only five distinct eigenvalues, by Lemma 2, the energy of the vertices of G_1 can be obtained by solving three 5×5 systems of equations, one each for a symmetry, from calculating the first moments on one side and directly counting walks on the other, considering the following three cases.

Case 1: Let $p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$ be the weights of the vertex $v \in V_1$ in G_1. Then, we have

$$\begin{aligned}
p_{11} + p_{12} + p_{13} + p_{14} + p_{15} &= 1\\
p_{11}\lambda_1 + p_{12}\lambda_2 + p_{13}\lambda_3 + p_{14}\lambda_4 + p_{15}\lambda_5 &= 0\\
p_{11}\lambda_1^2 + p_{12}\lambda_2^2 + p_{13}\lambda_3^2 + p_{14}\lambda_4^2 + p_{15}\lambda_5^2 &= 2\\
p_{11}\lambda_1^3 + p_{12}\lambda_2^3 + p_{13}\lambda_3^3 + p_{14}\lambda_4^3 + p_{15}\lambda_5^3 &= 2\\
p_{11}\lambda_1^4 + p_{12}\lambda_2^4 + p_{13}\lambda_3^4 + p_{14}\lambda_4^4 + p_{15}\lambda_5^4 &= 10.
\end{aligned}$$

Solving the system of equations, we get

$$p_{11} = 0.1, p_{12} = 0.1667, p_{13} = 0.0667, p_{14} = 0.3333, p_{15} = 0.3333.$$

Thus, the energy of each vertex $v \in V_1$ is given by

$$\mathcal{E}_{G_1}(v) = \sum_{j=1}^{n} p_{1j}\,|\lambda_j| = 1.0667.$$

Case 2: Let $p_{21}, p_{22}, p_{23}, p_{24}, p_{25}$ be the weights of any vertex $v \in V_2$ in G_1. Then, we have

$$\begin{aligned}
p_{21} + p_{22} + p_{23} + p_{24} + p_{25} &= 1\\
p_{21}\lambda_1 + p_{22}\lambda_2 + p_{23}\lambda_3 + p_{24}\lambda_4 + p_{25}\lambda_5 &= 0\\
p_{21}\lambda_1^2 + p_{22}\lambda_2^2 + p_{23}\lambda_3^2 + p_{24}\lambda_4^2 + p_{25}\lambda_5^2 &= 3\\
p_{21}\lambda_1^3 + p_{22}\lambda_2^3 + p_{23}\lambda_3^3 + p_{24}\lambda_4^3 + p_{25}\lambda_5^3 &= 4\\
p_{21}\lambda_1^4 + p_{22}\lambda_2^4 + p_{23}\lambda_3^4 + p_{24}\lambda_4^4 + p_{25}\lambda_5^4 &= 17.
\end{aligned}$$

Solving the system of equations, we get

$$p_{21} = 0.1, p_{22} = 1.1667, p_{23} = 0.15, p_{24} = 0.5833, p_{25} = 0.$$

Thus, for each of the vertices $v \in V_2$,

$$\mathcal{E}_{G_1}(v) = \sum_{j=1}^{n} p_{2j} \, |\lambda_j| = 1.5667.$$

Case 3: Let $p_{31}, p_{32}, p_{33}, p_{34}, p_{35}$ be the weights of any vertex $v \in V_3$ in G_1. Then, we have

$$\begin{aligned}
p_{31} + p_{32} + p_{33} + p_{34} + p_{35} &= 1 \\
p_{31}\lambda_1 + p_{32}\lambda_2 + p_{33}\lambda_3 + p_{34}\lambda_4 + p_{35}\lambda_5 &= 0 \\
p_{31}\lambda_1^2 + p_{32}\lambda_2^2 + p_{33}\lambda_3^2 + p_{34}\lambda_4^2 + p_{35}\lambda_5^2 &= 4 \\
p_{31}\lambda_1^3 + p_{32}\lambda_2^3 + p_{33}\lambda_3^3 + p_{34}\lambda_4^3 + p_{35}\lambda_5^3 &= 4 \\
p_{31}\lambda_1^4 + p_{32}\lambda_2^4 + p_{33}\lambda_3^4 + p_{34}\lambda_4^4 + p_{35}\lambda_5^4 &= 28.
\end{aligned}$$

Solving the system of equations, we get

$$p_{31} = 0.4, p_{32} = 0, p_{33} = 0.2667, p_{34} = 0, p_{35} = 0.3333.$$

Thus, the energy of each of the vertices $v \in V_3$ is given by

$$\mathcal{E}_{G_1}(v) = \sum_{j=1}^{n} p_{3j} \, |\lambda_j| = 1.6.$$

□

Note:

- As discussed in [11], the computation of the vertex energies of a graph G, with k distinct eigenvalues and r vertex symmetries, can also be done by combining the resultant r systems, each with k equations and k unknowns, together as a linear system

$$JP = Y,$$

where

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix}_{k \times k}, P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{r1} \\ p_{12} & p_{22} & \cdots & p_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1k} & p_{2k} & \cdots & p_{rk} \end{pmatrix}_{k \times r}$$

$$\text{and} \quad Y = \begin{pmatrix} \phi_1(1) & \phi_2(1) & \cdots & \phi_r(1) \\ \phi_1(A) & \phi_2(A) & \cdots & \phi_r(A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(A^{k-1}) & \phi_2(A^{k-1}) & \cdots & \phi_r(A^{k-1}) \end{pmatrix}_{k\times r}.$$

Solving the system $JP = Y$ gives the values of p_{ij}'s, which in turn, can be used to find the individual vertex energies of the graph.

- The above method is particularly useful if the number of vertex symmetries in a graph is large.
- For the remaining part of the article, we employ the above method to compute the vertex energies of graphs.

Theorem 4. *For the graph G_2, with $V(G_2) = \{v_1, \cdots, v_7\}$,*

$$\mathcal{E}_{G_2}(v) = \begin{cases} 1.5667 & \text{if } v = v_1, v_3, v_4, v_6, \\ 1.0667 & \text{if } v = v_2, v_5, \\ 1.6 & \text{otherwise.} \end{cases}$$

Proof. The characteristic polynomial of G_2 is given by

$$P(G_2; \lambda) = -\lambda^7 + 10\lambda^5 + 4\lambda^4 - 21\lambda^3 - 4\lambda^2 + 12\lambda.$$

Accordingly, the spectrum of G_2 is given by $3, -1, 1^2, -2^2, 0$ with the distinct eigenvalues being $\lambda_1 = 3, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = -2$ and $\lambda_5 = 0$. Thus, the vertex set of G_2 can be partitioned into 3 sets, based on 3 different vertex symmetries, namely $\{v_1, v_3, v_4, v_6\}$, $\{v_2, v_5\}$ and $\{v_7\}$, so that all the vertices belonging to a set have the same energy.

Therefore, as discussed in the earlier part of the section, we have the matrices J of order 5×5, P and Y of order 5×3, given by

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_5 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_5^2 \\ \lambda_1^3 & \lambda_2^3 & \cdots & \lambda_5^3 \\ \lambda_1^4 & \lambda_2^4 & \cdots & \lambda_5^4 \end{pmatrix}, \; P = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \\ p_{14} & p_{24} & p_{34} \\ p_{15} & p_{25} & p_{35} \end{pmatrix} \text{ and}$$

$$Y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 2 & 4 \\ 2 & 0 & 4 \\ 17 & 10 & 28 \end{pmatrix},$$

where Y is computed by counting the number of $v_i - v_i$ walks of length $j = 1, \cdots, 4$ for each $i = 1, 2, 3$. Solving the system $JP = Y$, we get

$$P = \begin{pmatrix} 0.15 & 0.0667 & 0.2667 \\ 0.25 & 0 & 0 \\ 0.3333 & 0.3333 & 0 \\ 0.2667 & 0.2667 & 0.4 \\ 0 & 0.3333 & 0.3333 \end{pmatrix}.$$

Substituting the values of p_{ij}'s in Lemma 1, we arrive at the required result. □

Using the proof technique adopted in Theorem 2, we obtain the vertex energies of all the remaining graphs $G_3, \cdots, G_{13}$. We omit the proofs for these as they are repetitive and present only the obtained values of each parameter namely, the distinct eigenvalues, vertex symmetries, the matrices Y and P and the corresponding vertex energies of each of the graphs in Tables 1 and 2.

3 Conclusion

In this article, we have determined the vertex energies of all connected non-regular non-bipartite integral graphs with maximum vertex degree four by finding the distinct eigenvalues and vertex symmetries to analyze the contribution of energy of each vertex towards the total energy of the graph.

The following are some of the observations made based on the work carried out.

- The computation of vertex energies of the graphs involved is non-trivial as the whole process involves observing the vertex symmetries in the graph and finding the number of walks of different lengths from a vertex to itself.
- Dealing with graphs with non-integral eigenvalues is even more complex as it involves solving linear systems with rational/irrational coefficients.
- The study of vertex energy distribution of classes of graphs of general order is more challenging owing to the complexity involved. In this direction, the authors in [7] have proposed a general method to find

Table 1: Vertex energies of the graphs $G_3, \cdots, G_8$

Name of the graph	Distinct eigen values	No. of vertex symmetries	Y	P	Vertex Energies
G_3	-2 -1 0 3	2	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 6 & 2 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.2 \\ 0.5 & 0 \\ 0 & 0.6667 \\ 0.3 & 0.1333 \end{pmatrix}$	$\begin{cases} 1.8 & \text{if } v = v_1, v_2, \\ 0.8 & \text{otherwise.} \end{cases}$
G_4	-2 -1 0 1 2 3	5	$\begin{pmatrix} 2 & 4 & 4 & 4 & 2 & 0 \\ 10 & 17 & 28 & 22 & 4 & 4 \\ 18 & 38 & 52 & 28 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 4 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.1 & 0.1 & 0.4 & 0.4333 & 0.1083 \\ 0.3333 & 0.5833 & 0 & 0.0833 & 0.0833 \\ 0.3333 & 0 & 0.3333 & 0 & 0.5833 \\ 0 & 0.1667 & 0 & 0.1667 & 0.1667 \\ 0.0667 & 0.15 & 0.2667 & 0.15 & 0.0167 \end{pmatrix}$	$\begin{cases} 1.0667 & \text{if } v = v_1, \\ 1.5667 & \text{if } v = v_2, v_3, \\ 1.6 & \text{if } v = v_4, \\ 1.9 & \text{if } v = v_5, v_6, \\ 0.6 & \text{otherwise.} \end{cases}$
G_5	-2 -1 0 1 2 3	3	$\begin{pmatrix} 22 & 28 & 4 \\ 0 & 0 & 0 \\ 28 & 52 & 2 \\ 2 & 4 & 0 \\ 4 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.4333 & 0.4 & 0.1083 \\ 0.0833 & 0 & 0.0833 \\ 0 & 0.3333 & 0.5833 \\ 0.1667 & 0 & 0.1667 \\ 0.1667 & 0 & 0.0417 \\ 0.15 & 0.2667 & 0.0167 \end{pmatrix}$	$\begin{cases} 1.9 & \text{if } v = v_2, v_3, v_8, v_{10}, \\ 1.6 & \text{if } v = v_7, \\ 0.6 & \text{otherwise.} \end{cases}$
G_6	-2 -1 0 1 3	4	$\begin{pmatrix} 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 10 & 28 & 17 & 3 \\ 2 & 4 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.2667 & 0.4 & 0.2667 & 0.0667 \\ 0 & 0 & 0.25 & 0.25 \\ 0.3333 & 0.3333 & 0 & 0.3333 \\ 0.3333 & 0 & 0.3333 & 0.3333 \\ 0.0667 & 0.2667 & 0.15 & 0.0167 \end{pmatrix}$	$\begin{cases} 1.0667 & \text{if } v = v_1, v_2, \\ 1.6 & \text{if } v = v_3, v_4, \\ 1.5667 & \text{if } v = v_5, v_7, \\ 0.7667 & \text{otherwise.} \end{cases}$
G_7	-2 -1 1 2 3	3	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 8 & 24 & 19 \\ 2 & 4 & 3 \\ 2 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.0333 & 0.3 & 0.1333 \\ 0.5833 & 0.3333 & 0.5 \\ 0.1667 & 0 & 0.1667 \\ 0.1667 & 0.1667 & 0.0 \\ 0.05 & 0.2 & 0.2 \end{pmatrix}$	$\begin{cases} 1.3 & \text{if } v = v_1, v_2, v_6, v_7, \\ 1.8667 & \text{if } v = v_3, v_5, \\ 1.5333 & \text{otherwise.} \end{cases}$
G_8	-2 -1 1 3	3	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.2 & 0.1333 & 0.4667 \\ 0.25 & 0.5 & 0 \\ 0.5 & 0.1667 & 0.3333 \\ 0.05 & 0.2 & 0.2 \end{pmatrix}$	$\begin{cases} 1.3 & \text{if } v = v_1, v_2, v_3, v_4, \\ 1.5333 & \text{if } v = v_5, v_7, \\ 1.8667 & \text{otherwise.} \end{cases}$

Table 2: Vertex energies of the graphs $G_9, \cdots, G_{13}$

Name of the graph	Distinct eigen values	No. of vertex symmetries	Y	P	Vertex Energies
G_9	-2 -1 0 1 2 3	5	$\begin{pmatrix} 4 & 2 & 2 & 2 & 0 \\ 17 & 18 & 22 & 10 & 4 \\ 38 & 28 & 28 & 18 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 4 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.1 & 0.3083 & 0.4333 & 0.1 & 0.1083 \\ 0.5833 & 0.0833 & 0.0833 & 0.3333 & 0.0833 \\ 0 & 0.25 & 0 & 0.3333 & 0.5833 \\ 0.1667 & 0.0417 & 0.1667 & 0 & 0.1667 \\ 0.15 & 0.15 & 0.15 & 0.0667 & 0.0167 \end{pmatrix}$	$\begin{cases} 1.5667 & \text{if } v = v_1, v_3, \\ 1.4 & \text{if } v = v_2, v_4, \\ 1.9 & \text{if } v = v_5, v_6, \\ 1.0667 & \text{if } v = v_7, \\ 0.6 & \text{otherwise.} \end{cases}$
G_{10}	-2 -1 0 1 2 3	3	$\begin{pmatrix} 22 & 10 & 10 \\ 0 & 0 & 0 \\ 28 & 18 & 8 \\ 2 & 2 & 0 \\ 4 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.4333 & 0.1 & 0.2667 \\ 0.0833 & 0.3333 & 0 \\ 0 & 0.3333 & 0.3333 \\ 0.1667 & 0.1667 & 0.3333 \\ 0.15 & 0.0667 & 0.0667 \end{pmatrix}$	$\begin{cases} 1.9 & \text{if } v = v_2, v_4, v_6, v_8, \\ 1.0667 & \text{otherwise.} \end{cases}$
G_{11}	-2 -1 0 1 2 3	4	$\begin{pmatrix} 22 & 10 & 4 & 18 \\ 0 & 0 & 0 & 0 \\ 28 & 8 & 2 & 28 \\ 2 & 0 & 0 & 2 \\ 4 & 2 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.4333 & 0.2667 & 0.1083 & 0.3083 \\ 0.0833 & 0 & 0.0833 & 0.0833 \\ 0 & 0.3333 & 0.5833 & 0.25 \\ 0.1667 & 0.3333 & 0.1667 & 0.1667 \\ 0.1667 & 0 & 0.0417 & 0.0417 \\ 0.15 & 0.0667 & 0.0167 & 0.15 \end{pmatrix}$	$\begin{cases} 1.9 & \text{if } v = v_1, v_3, v_4, v_8, \\ 1.0667 & \text{if } v = v_2, v_5, v_6, \\ 0.6 & \text{if } v = v_7, v_{10}, v_{11}, \\ 1.4 & \text{otherwise.} \end{cases}$
G_{12}	-2 -1 0 1 2 3	4	$\begin{pmatrix} 13 & 9 & 24 & 8 \\ 3 & 2 & 4 & 2 \\ 6 & 6 & 34 & 16 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.3667 & 0.2417 & 0.4667 & 0.0333 \\ 0.0833 & 0.0833 & 0 & 0.5833 \\ 0 & 0.25 & 0 & 0 \\ 0.3333 & 0.3333 & 0.3333 & 0.1667 \\ 0.1667 & 0.0417 & 0 & 0.1667 \\ 0.05 & 0.05 & 0.2 & 0.05 \end{pmatrix}$	$\begin{cases} 1.6333 & \text{if } v = v_1, v_9, \\ 1.1333 & \text{if } v = v_2, v_8, v_{10}, v_{11}, \\ 1.8667 & \text{if } v = v_3, v_4, v_7, \\ 1.3 & \text{otherwise.} \end{cases}$
G_{13}	-2 0 1 2 3	4	$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 13 & 9 & 22 & 18 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.3833 & 0.2583 & 0.45 & 0.325 \\ 0.1667 & 0.4167 & 0.1667 & 0.4167 \\ 0.1667 & 0.1667 & 0 & 0 \\ 0.25 & 0.125 & 0.25 & 0.125 \\ 0.0333 & 0.0333 & 0.1333 & 0.1333 \end{pmatrix}$	$\begin{cases} 1.5333 & \text{if } v = v_1, v_9, \\ 1.0333 & \text{if } v = v_2, v_7, v_8, v_{10}, \\ 1.8 & \text{if } v = v_3, v_6, \\ 1.3 & \text{otherwise.} \end{cases}$

the vertex energies of any graph by finding all its eigenvalues, the corresponding eigenvectors and then finding the orthogonal eigenvector matrix. This method is particularly applicable for general graphs with larger vertex degrees and with less/no symmetries.

- The energy distribution among the vertices in the graphs does not seem to correlate with the structural properties of the graph such as vertex degree, centrality and distance. For instance, the vertices v_4 and v_5 in G_4 have the same vertex degree but different vertex energies. Similarly, in G_5, the central vertex v_7 has lesser energy compared to v_2 whereas in G_{11}, the energy of one of the central vertices v_1 is greater than that of the pendent vertices. Also, in G_{10}, vertices in two different vertex symmetry sets, namely $\{v_1, v_{10}\}$ and $\{v_3, v_5, v_7, v_9\}$, have the same vertex energy. Thus, vertices that do not belong to the same vertex symmetry set may have the same vertex energy.

These observations imply that the study of energy distribution of vertices in a graph is very interesting and challenging, particularly for general classes of graphs. The method used in the study may further be extended to define and determine the vertex energies of directed graphs, weighted graphs, fuzzy graphs, etc. which provides a new direction of research. It can be therefore concluded that there is a lot of scope for further research work in this field of study.

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Author contributions

PNS, SBP, NN: Conceptualization, methodology, and writing - original draft; NHM, VCKU: Formal analysis and validation.

Data Availability Statement

The data supporting this work are cited within the text as references.

Declarations

Ethical Approval Not applicable.

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References

[1] Octavio Arizmendi, Jorge Fernandez Hidalgo, and Oliver Juarez-Romero. Energy of a vertex. *Linear Algebra and its Applications*, 557:464–495, 2018. doi:10.1016/j.laa.2018.08.014.

[2] Octavio Arizmendi and Oliver Juárez Romero. On bounds for the energy of graphs and digraphs. In Fernando Galaz-García, Juan Carlos Pardo Millán, and Pedro Solórzano, editors, *Contributions of Mexican Mathematicians Abroad in Pure and Applied Mathematics*, volume 709, pages 1–19. American Mathematical Society, 2018. doi:10.1090/conm/709/14288.

[3] Krystyna T. Baliǹska, Dragoš Cvetković, Zoran Radosavljević, Slobodan Simić and Dragan Stevanović. A survey on integral graphs. *Publikacije Elektrotehničkog fakulteta. Serija Matematika*, (13):42–65, 2002. URL: http://www.jstor.org/stable/43666564.

[4] Andries E. Brouwer. Small integral trees. *The Electronic Journal of Combinatorics*, 15(1), 2008. URL: http://eudml.org/doc/129709.

[5] Andries E. Brouwer and Wilhelmus H. Haemers. *Spectra of Graphs*. Springer New York, 2012. doi:10.1007/978-1-4614-1939-6.

[6] Ivan Gutman. The energy of a graph. *Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz*, 103:1–22, 1978.

[7] Ivan Gutman and Boris Furtula. Calculating vertex energies of graphs - A tutorial. *MATCH Communications in Mathematical and in Computer Chemistry*, 93(3):691–698, 2025. doi:10.46793/match.93-3.691G.

[8] Frank Harary and Allen J. Schwenk. Which graphs have integral spectra? In Ruth A. Bari and Frank Harary, editors, *Graphs and*

Combinatorics, pages 45–51. Springer Berlin Heidelberg, 1974. doi:10.1007/BFb0066434.

[9] Zoran Radosavljevic and Slobodan Simic. There are just thirteen connected nonregular nonbipartite integral graphs having maximum vertex degree four (a shortened report). In *Proceedings Sixth Yugoslav Seminar on Graph Theory (Dubrovnik 1985), University of Novi Sad*, pages 183–187, 1986.

[10] Harishchandra S. Ramane, Sheena Y. Chowri, Thippeswamy Shivaprasad, and Ivan Gutman. Energy of vertices of subdivision graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, 93(3):701–711, 2025. doi:10.46793/match.93-3.701R.

[11] Hunjanalu T. Sharathkumar, Chikkenahalli K. Shrikanth, Narahari N. Swamy, Hadonahally M. Nagesh, and Uppara Vijay Chandra Kumar. On the vertex energy of small integral trees. *MATCH Communications in Mathematical and in Computer Chemistry*, 95(2):467–485, 2026. doi:10.46793/match.95-2.10325.

[12] Chikkenahalli K. Shrikanth, Hunjanalu T. Sharathkumara, Narahari N. Swamy, Hadonahally M. Nagesh, and Uppara Vijay Chandra Kumar. Vertex energy of small integral graphs. *Boletim da Sociedade Paranaense de Matemática*, 43:1–14, 2025. doi:10.5269/bspm.77666.

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